Exercise 98

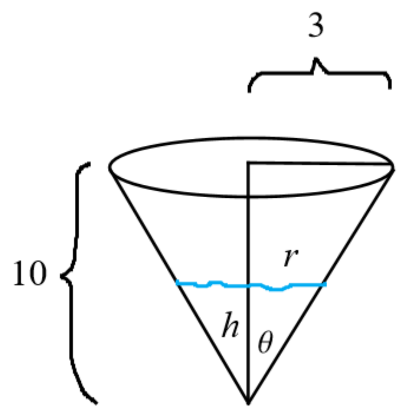
A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

Solution

The water takes the shape of a cone as it pours into the cup, so its volume is

$$V = \frac{1}{3}\pi r^2 h.$$

The aim is to find dh/dt when $h = 5$, so eliminate r in favor of h by using the geometry of the cup.



$$\tan\theta = \frac{r}{h} = \frac{3}{10} \quad \to \quad r = \frac{3}{10}h$$

As a result, the volume becomes

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(\frac{3}{10}h\right)^2 h \\ &= \frac{1}{3}\pi\left(\frac{9}{100}h^2\right) h \\ &= \frac{3\pi}{100}h^3. \end{aligned}$$

Differentiate both sides with respect to t by using the chain rule.

$$\begin{aligned} \frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{3\pi}{100}h^3\right) \\ \frac{dV}{dt} &= \frac{3\pi}{100}(3h^2)\cdot\frac{d}{dt}(h) \end{aligned}$$

Set $dV/dt = 2$ and $h = 5$ and solve for dh/dt.

$$\frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt}$$

$$2 = \frac{9\pi}{100}(5)^2\frac{dh}{dt}$$

Therefore, the rate at which the water level is rising when the water is 5 cm deep is

$$\frac{dh}{dt} = \frac{8}{9\pi} \approx 0.282942\ \frac{\text{cm}}{\text{s}}.$$